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Is the Approximation of a Function by Its Fejér Means Monotone?

LEONARDO COLZANI

Dipartimento di Matematica, Universita' degli Studi di Milano, via C. Saldini, 50, 20133 Milano, Italy

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Let f be a function in $L^p(\mathbf{T})$, $1 \leqslant p < +\infty$, or let f be a continuous function on the torus T if $p = +\infty$, and let K_n be the nth Fejér kernel. We prove that, although the sequence $\{\|f - K_n^* f\|_p\}$ is not monotone in general, it still has a monotonicity property. Namely, if $m < n$, then $\|f - K_n^* f\|_p \leqslant (2 + m/n)^{|2/p - 1|} \|f - K_m^* f\|_p$. © 1989 Academic Press, Inc.

Let f be an integrable function on the torus $\mathbf{T}$ $(= [-\frac{1}{2}, \frac{1}{2}))$ and let $K_n = \sum_{|j| \leqslant n} (1 - |j|/n) \exp(2\pi i j \circ)$ be the nth Fejér kernel. It is well known that the convolution $K_n^* f$ is an approximation of f. In particular, when f is in $L^2(\mathbf{T})$,

$$\|f - K_n^* f\|_2 = \left\{ \sum_{|j| < +\infty} \min(|j|^2/n^2, 1)\, |\hat{f}(j)|^2 \right\}^{1/2},$$

and the sequence $\{\|f - K_n^* f\|_2\}$ is monotone decreasing and $\lim_{n \to +\infty} \|f - K_n^* f\|_2 = 0$. When f is in $L^p(\mathbf{T})$ $(1 \leqslant p < +\infty)$ or when f is continuous on $\mathbf{T}$ $(p = +\infty)$, it is still true that $\lim_{n \to +\infty} \|f - K_n^* f\|_p = 0$; however, in some cases the approximation of f by $K_n^* f$ can be worse than the approximation of f by $K_m^* f$, $m < n$, i.e., the sequence $\{\|f - K_n^* f\|_p\}$ is not always monotone. For example, the following theorem holds.

THEOREM 1. *Given $\varepsilon > 0$ and m, there exists $n > m$ and a continuous function f such that*

$$\|f - K_n^* f\|_\infty > (2 - \varepsilon) \|f - K_m^* f\|_\infty.$$

Despite this negative result we shall prove that the sequence $\{\|f - K_n^* f\|_p\}$ still has the monotonicity property expressed by the following theorem.

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THEOREM 2. *For every function f in $L^p(\mathbf{T})$ $(1 \leqslant p < +\infty)$ or continuous on $\mathbf{T}$ $(p = +\infty)$, and every $n > m$, we have*

$$\|f - K_n^* f\|_p \leqslant (2 + m/n)^{|2/p - 1|} \|f - K_m^* f\|_p.$$

It is convenient to prove Theorem 2 first.

Proof of Theorem 2. Note that

$$(f - K_n^* f)\hat{\ }(j) = \varphi_{m,n}(j)(f - K_m^* f)\hat{\ }(j),$$

where

$$\begin{aligned} \varphi_{m,n}(j) &= m/n && \text{if } \ |j| < m, \\ &= |j|/n && \text{if } \ m \leqslant |j| < n, \\ &= 1 && \text{if } \ n \leqslant |j|. \end{aligned}$$

Also note that the multiplier $\varphi_{m,n}$ is the Fourier transform of the measure

$$\Phi_{m,n} = \delta_0 - K_n + m/nK_m,$$

where δ_0 denotes the unit mass measure at 0. Hence

$$f - K_n^* f = \Phi_{m,n}^*(f - K_m^* f),$$

and since the total variation of the measure $\Phi_{m,n}$ is less than $2 + m/n$, the theorem follows in the cases $p = 1$ and $p = +\infty$. The case $p = 2$ is an immediate consequence of the Plancherel formula, and the other cases easily follow by interpolation. ∎

Proof of Theorem 1. The idea is that for a fixed m it is possible to choose n so large that the total variation of the measure $\Phi_{m,n}$ is greater than $2 - \varepsilon$. Let g be a continuous piecewise linear function such that

$$\begin{aligned} g(x) &= 1 && \text{if } \ x = 0, \\ &= -1 && \text{if } \ \delta \leqslant |x| \leqslant \tfrac{1}{4}, \\ &= 1 && \text{if } \ \tfrac{1}{4} + \delta \leqslant |x| \leqslant \tfrac{1}{2}, \end{aligned}$$

where δ is a very small positive number. Note that $\|g\|_\infty = 1$ and that g has mean zero. It is easy to check that if δ is very small and n is very large, then $\Phi_{m,n}^* g(0) > 2 - \varepsilon$. Define now f by

$$\begin{aligned} \hat{f}(j) &= m/|j| \ \hat{g}(j) && \text{if } \ 0 < |j| < m, \\ &= \hat{g}(j) && \text{if } \ m \leqslant |j|. \end{aligned}$$

and $\hat{f}(0)$ arbitrary. Then

$$f - K_m^* f = g,$$

and

$$\begin{aligned} \|f - K_n^* f\|_\infty &= \|\Phi_{m,n}^* g\|_\infty \\ &\geq (2-\varepsilon)\|g\|_\infty \\ &= (2-\varepsilon)\|f - K_m^* f\|_\infty . \quad \blacksquare \end{aligned}$$

We conclude this note by observing that an analogue of Theorem 1 holds for every family of approximations of the identity formed by trigonometric polynomials. However, it is not difficult to construct families of approximations of the identity for which no analogue of Theorem 2 is true.

Note added in proof. Professor L. De Michele has obtained an extension of Theorem 1 to $L^p(\mathbf{T})$, $1 \leq p < +\infty$ and $p \neq 2$.